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LETTER TO THE EDITOR

Dynamic versus kinematic symmetry breaking in a two-dimensional model of collective motion

J Cseh

Institute of Nuclear Research, Hungarian Academy of Sciences, PO Box 51, Debreen, Hungary 4001†, and Institut für Theoretische Physik, Justus-Liebig-Universität, Giessen, Federal Republic of Germany.

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Abstract. We investigate the question of whether or not the vibrational spectrum of a two-dimensional interacting boson model can be transformed into the rotational one via quantum deformation of the group structure. It turns out that although the spectrum of the q -deformed model shows one or another of the rotational-like features at different values of the deformation parameter, nevertheless, all the requirements of the rotational behaviour cannot be obtained simultaneously. This result indicates that the quantum algebraic treatment cannot give an analytical solution to the eigenvalue problem of the interacting boson models of nuclear and molecular physics in the whole range of the physically relevant breaking of dynamical symmetries.

In addition to the Lie algebraic description of the collective motion of many-body systems [1–4], more recently another type of algebraic approach has been proposed based on quantum groups [5]. From the mathematical point of view the q -deformed groups are Hopf algebras [6], containing the classical Lie algebras as limiting cases. One of the consequences of the more general algebraic structure of these models is that a description in terms of dynamical symmetry is applicable to a larger territory than a similar description of the Lie algebraic models. The concept of the dynamical symmetry, as discussed in more detail below, plays a crucial role in the algebraic approach, e.g. it provides us with an analytical solution of the eigenvalue problem. At present, however, it is not clear, how far this concept can be extended by making use of quantum groups. Whether or not the whole range of the physically relevant breaking of the dynamical symmetry in the classical Lie algebraic models can be covered by the analytical solutions of the quantum algebraic treatment. For example, can a vibrational spectrum of the Lie algebraic model be transformed into a rotational one via q -deformation of the algebraic structure? The present letter addresses this question in the framework of a two-dimensional interacting boson model of collective motion [7] and its quantum extension [8]. In [9] a similar problem has been investigated in a simpler form. There the energy spectra of the one-dimensional $SU(2)$ and $SU_q(2)$ models were considered. Via the q -deformation of the vibrational limit some rotational-like features, e.g. degeneracies of the one-dimensional spectrum have been found; however, the exact rotational spectrum was not obtained. Here we

† Permanent address.

consider both the energy spectra and the electric transitions of the $SU(3)$ and $SU_q(3)$ models. Such an investigation can give a hint of the possible range of applicability of the analytically solvable q -deformed algebraic models of problems with higher dimensions, namely about the proposed [5,8] quantum algebraic extensions of the $SU(4)$ vibron model and $SU(6)$ quadrupole interacting boson model.

In the Lie algebraic description of the quadrupole motion of atomic nuclei [1], vibrational-rotational motion of molecules [2], clusterization of nuclei [3], or structure of hadrons [4], the spectrum of the system is generated by a dynamical algebra, which usually has more than one subalgebra. The physically relevant subalgebras contain the algebra of angular momentum, therefore one obtains chains of nested subalgebras:

$$\begin{aligned} G &\supset G'_1 \supset G''_1 \supset \dots \\ G &\supset G'_2 \supset G''_2 \supset \dots \\ G &\supset G'_n \supset G''_n \supset \dots \end{aligned} \quad (1)$$

Each of them provides us with a complete set of basis states carrying the representation labels of the algebras as quantum numbers. Furthermore, when the Hamiltonian of the system can be expressed as the function of the Casimir operators of a single chain of algebras the eigenvalue problem has an analytical solution. In such a case the system is said to have a dynamical symmetry. These dynamical symmetries correspond to simple geometrical and dynamical features of the system, e.g. rotation of a rigid rotor with permanent deformation, or soft vibration around a spherical equilibrium shape, etc. For some systems the description based on the dynamical symmetry proved to be a good approximation. In most of the cases, however, the real physical system corresponds to a situation in between these limiting cases. Then the matrix elements of the generators of more than one algebra chain have to be calculated, and the eigenvalue problem is solved by a numerical diagonalization. With this procedure, which we call dynamic breaking of the symmetry, one can go smoothly from one limiting case of the model to the other.

As for the applications of quantum algebraic models to nuclear and molecular physics, they have been concentrated so far on the $SU_q(2)$ group, and its boson realisation in terms of oscillator quanta [5,6]. This model is a q -deformed version of the interacting boson model with the dynamical algebra of the $SU(2)$ group, which describes one-dimensional motions. When the deformation parameter is equal to zero, the Hopf algebraic description goes back to the classical Lie algebraic model. For non-zero values the symmetry is broken kinematically, i.e. via the deformation of the commutation relations of the operators. (This kind of kinematically broken Lie algebraic dynamical symmetry can be considered as a more general quantum algebraic dynamical symmetry.) It is important from the present point of view, that in applications to nuclear and molecular spectra the deformation parameter has values close to zero [5].

Here we are interested in the essential breaking of the symmetries of the Lie algebraic models. In particular, we would like to know whether the kinematic breaking of symmetry can take us from one limit of the model (e.g. vibrational) to the other (rotational) and similarly with dynamic symmetry breaking of the Lie algebraic description.

In the two-dimensional interacting boson model the building blocks are bosons with angular momentum $m = 0 \pm 2$. Their creation (a_0^\dagger , a_+^\dagger , a_-^\dagger) and annihilation operators (a_0 , a_+ , a_-) satisfy the usual boson commutation relations:

$$[a_i, a_j^\dagger] = \delta_{ij} \quad [a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0 \quad (2)$$

and the physical operators are obtained in terms of their particle number conserving bilinear products

$$\Lambda_{ij} = a_i^\dagger a_j \quad (3)$$

which satisfy the

$$[\Lambda_{ij}, \Lambda_{kl}] = \delta_{jk} \Lambda_{il} - \delta_{il} \Lambda_{kj} \quad (4)$$

commutation relations, and generate the U(3) group. (Here we follow the notations of most of the papers related to the interacting boson model, and refer to the groups rather than to their algebras.) The

$$N = \sum_i \Lambda_{ii} \quad (5)$$

particle number operator is kept constant, therefore the number of independent operators is eight, and the underlying group structure of the model is SU(3). There are two group chains containing the angular momentum group SO(2). The first one describes the vibrational motion

$$\begin{array}{l} \text{SU}(3) \supset \text{SU}(2) \supset \text{SO}(2) \\ | N, \quad \quad \quad n_d, \quad \quad M \rangle \\ n_d = N, N-1, \dots, 1, 0 \quad M/2 = n_d, n_d-2, \dots, 1 \text{ or } 0 \end{array} \quad (6)$$

and the second one

$$\begin{array}{l} \text{SU}(3) \supset \text{SO}(3) \supset \text{SO}(2) \\ | N, \quad \quad \quad l, \quad \quad M \rangle \\ l = N, N-2, \dots, 1, \text{ or } 0 \quad M/2 = l, l-1, \dots, 1, 0 \end{array} \quad (7)$$

characterizes the rotational limit. When the dynamical symmetry (6) holds, the energy eigenvalue is given by

$$E_v = A + Bn_d + Cn_d^2 + DM^2 \quad (8)$$

while the allowed E2 transitions are

$$\langle N, n_d + 1, M \pm 2 | Q_\pm | N, n_d, M \rangle = \sqrt{(N - n_d)(n_d/2 \pm M/4 + 1)} \quad (9a)$$

and

$$\langle N, n_d - 1, M \pm 2 | Q_\pm | N, n_d, M \rangle = \sqrt{(N - n_d + 1)(n_d/2 \mp M/4)} \quad (9b)$$

Here

$$Q_\pm = a_\pm^\dagger a_0 + a_0^\dagger a_\mp \quad (9c)$$

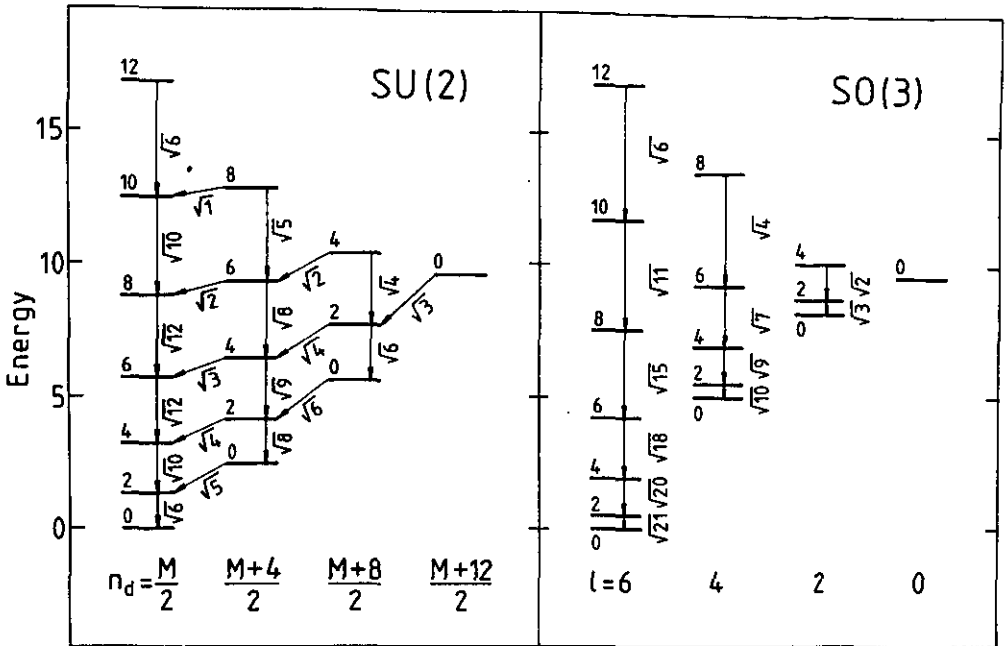


Figure 1. Vibrational (SU(2)) and rotational (SO(3)) spectra of the two-dimensional interacting boson model with $N = 6$ total number of bosons. The angular momentum values (M) are given above the states, and the bands are characterized by the quantum numbers shown at the bottom of the figure. The arrows indicate electric transitions, and their relative strengths are given by the numbers under the square roots.

is the two-dimensional quadrupole transition operator. For the rotational limit of (7) the corresponding analytical formulae are

$$E_r = a + bl(l + 1) + dM^2 \tag{10}$$

$$\langle N, l, M \pm 2 | Q_{\pm} | N, l, M \rangle = \frac{1}{\sqrt{2}} \sqrt{(l \mp M/2)(l \pm M/2 + 1)} . \tag{11}$$

Typical vibrational and rotational spectra are shown in the left- and right-hand sides of figure 1, respectively. The parameters of the energy formulae are $A = 0.0$, $B = 1.0$, $C = 0.1$, $D = 0.05$, and $a = 9.6$, $b = -0.2286$, $d = 0.1167$. The strengths of the reduced electric transitions are also shown in arbitrary units. The interband transitions disappear in the rotational limit.

In the quantum-deformed algebraic description the

$$A_{ij} = a_i^\dagger a_j \tag{12}$$

operators close under the q -commutation relations [8], defined as:

$$[A, B]_q = AB - qBA . \tag{13}$$

The group chain corresponding to the vibrational limit of (6) also exists in the q -algebraic model:

$$SU_q(3) \supset SU_q(2) \supset SO_q(2) \tag{14}$$

while the analogue of (7) does not. The matrix elements giving the energy eigenvalues and the transitions for the group chain (13) can also be obtained analytically [6, 8]:

$$E_q = A + B[n_d] + C[n_d][n_d] + D[M][M] \quad (15)$$

and

$$\langle N, n_d + 1, M \pm 2 | Q_{\pm} | N, n_d, M \rangle_q = \sqrt{[N - n_d][n_d/2 \pm M/4 + 1]} \quad (16a)$$

$$\langle N, n_d - 1, M \pm 2 | Q_{\pm} | N, n_d, M \rangle_q = \sqrt{[N - n_d + 1][n_d/2 \mp M/4]} \quad (16b)$$

where

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}}. \quad (17)$$

For the q -boson realisations of $SU_q(3)$ used in [8], two distinct cases of the q -deformation are important for physical applications:

(i) When q is real ($q = e^{\tau}$, with τ real) the q -numbers can be written as

$$[x] = \frac{\sinh \tau x}{\sinh \tau}. \quad (18)$$

(ii) When q is a phase ($q = e^{i\tau}$, with τ real) the q -numbers have the form

$$[x] = \frac{\sin \tau x}{\sin \tau}. \quad (19)$$

In light of these expressions the question raised in the introduction, whether the kinematic breaking of the vibrational symmetry can result in a rotational spectrum, requires calculations of energies and electric transitions in terms of equations (15) and (16). It turns out that the pure imaginary deformation parameter of equation (19) does not take us towards the rotational limit. For example the ratio of the energies of the 4^+ and 2^+ states in the ground-state band is 2.46 for the $\tau = 0$ vibrational limit, with the parameters of figure 1, and it is 4 for the rotational case (this latter value is independent of the parameters of equation (10), if the ground state is at zero energy, as in figure 1). When q is a phase, this ratio decreases with increasing τ values, contrary to the requirement of the rotational features.

For real values of q the deformation of the spectrum takes place in the right direction. The above-mentioned energy ratio increases with increasing τ , the ratio of the interband and intraband transitions decreases, etc. However, the real rotational spectrum could not be obtained with this kind of deformation either, because the different characteristics of the rotational behaviour do not show up at the same value of the deformation parameter. Figure 2 illustrates the situation. For the τ value which gives the correct energy ratio of the lowest-lying 4^+ and 2^+ states, the other members of the band, e.g. the 12^+ state, appear at a far too high energy, and the ratio of the $B(E2)$ values for the interband and intraband transitions among the lowest-lying states

$$\frac{B(E2, 0_2^+ \rightarrow 2_1^+)}{B(E2, 2_1^+ \rightarrow 0_1^+)} \quad (20)$$

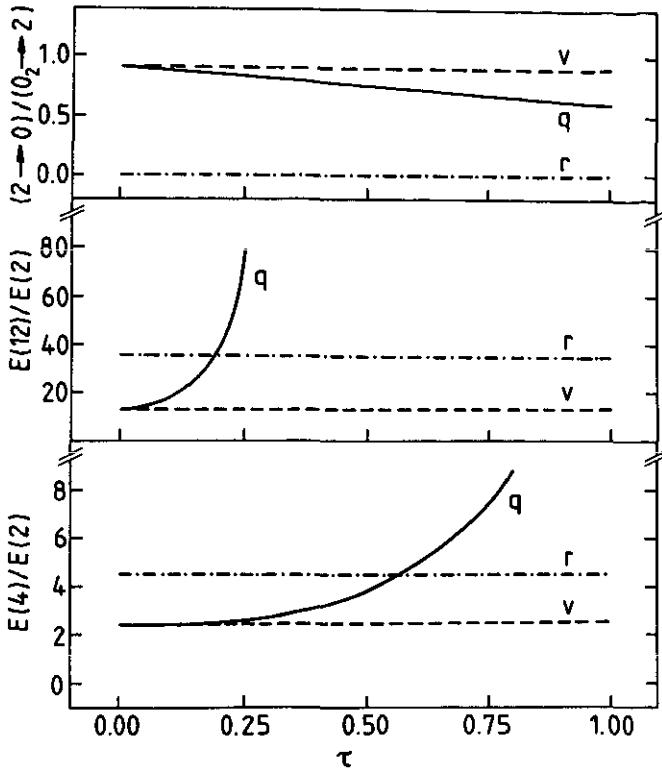


Figure 2. Characteristic ratios of energies and electric transitions of the two-dimensional algebraic collective models. The values corresponding to the vibrational and rotational limits of the classical Lie algebraic description are denoted by v and r , respectively. The curves labelled with q show the results of the quantum algebraic model, as a function of the real deformation parameter ($q = e^\tau$). The lower and middle panels display two energy ratios of the ground state band, while the upper part shows the $B(E2)$ ratios of the interband $0_2^+ \rightarrow 2_1^+$ and intraband $2_1^+ \rightarrow 0_1^+$ electric transitions.

is also very large. This means that via the quantum deformation of the algebraic structure (6) which describes the vibrational motion, one cannot reach the rotational spectrum of the two-dimensional model.

Relating to some applications of the $SU_q(2)$ algebraic model to molecular and nuclear spectra, it was pointed out [5] that the quantum algebraic description is able to sum up in closed formulae certain series expansions, which appear in the calculation of expectation values of physical operators. The solution to the eigenvalue problem of a one-dimensional collective motion has been written analytically, and it was found to be in close agreement with the behaviour of some physical systems in which the dynamical symmetry, in the classical Lie algebraic sense, is broken. Here we have shown in the example of the two-dimensional Lie and Hopf algebraic models that the gap between the dynamical symmetries of the Lie algebraic description is not bridged by the quantum deformation of the corresponding group chain, i.e. via the kinematic breaking of the symmetry. The quantum algebraic treatment extends the applicability of the analytical solutions; however, it seems to work only in the near neighbourhood of the exact dynamical symmetry. This finding indicates that the proposed [6,8]

quantum extensions of the Lie algebraic models with higher ranks (SU(4), SU(6), etc) provide us with analytical solutions in a limited range of the symmetry breaking, which might be treated in the traditional approach by perturbation calculations.

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